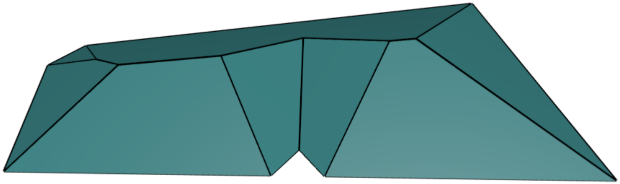
Bisector Graphs for Min-/Max-Volume Roofs

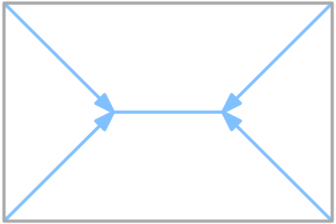
over Simple Polygons

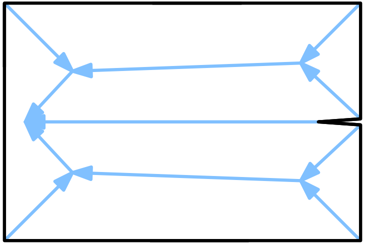
Günther Eder – Martin Held – Peter Palfrader

March 2016, Lugano



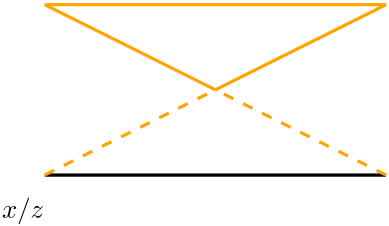
- Comparing two polygons. A lower area does not always lead to a lower roof volume.
- The lower envelope over all planes is not the minimum volume roof. (Neither does the upper envelope lead to the maximum volume roof.)





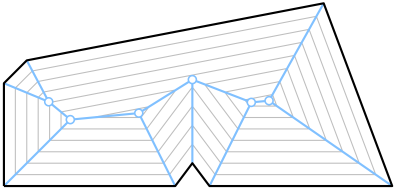
Motivation

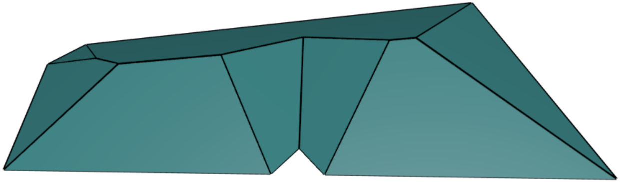
- Comparing two polygons. A lower area does not always lead to a lower roof volume.
- The lower envelope over all planes is not the minimum volume roof. (Neither does the upper envelope lead to the maximum volume roof.)



APPROACH

- Building on *Roof Model* and *Bisector Graphs*[2].
- *Gradient Property*[2] generalized.
- *Wavefront Propagation*[1] extended by two additional events.





2. Oswin Aichholzer, Franz Aurenhammer, David Alberts, and Bernd Gärtner. A Novel Type of Skeleton for Polygons. *Journal of Universal Computer Science*, 1995

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THEOREM (ROOF $\leftrightarrow$ BISECTOR GRAPH [2])

Every roof for P corresponds to a unique bisector graph of P, and vice versa.

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NATURAL GRADIENT PROPERTY

Let $\mathcal{R}(\mathcal{P})$ be a roof for $\mathcal{P}$. We say that a facet f of $\mathcal{R}(\mathcal{P})$ has the *natural gradient property* if, for every point $p \in f$, there exists a path that (i) starts at p, (ii) follows the steepest gradient, and (iii) reaches the boundary of $\mathcal{P}$.

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Approach

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Extended Wavefront Propagation

- Edge Event and Split Event[2].
- Create Event and Divide Event

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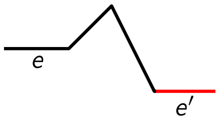
GENERAL POSITION

- No two edges of $\mathcal{P}$ are parallel to each other.
- Not more than three bisectors of edges of $\mathcal{P}$ meet in one point.



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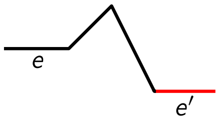

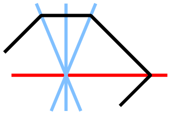

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DEFINITION (MIN-/MAX-VOLUME BISECTOR GRAPH)

The *maximum-volume bisector graph* $\mathcal{B}_{\text{max}}(\mathcal{P})$ of a polygon $\mathcal{P}$ is a bisector graph $\mathcal{B}(\mathcal{P})$ where the associated roof $\mathcal{R}(\mathcal{P})$ has the natural gradient property for each of its facets and that maximizes the volume over all possible natural roofs for $\mathcal{P}$. Similarly for the *minimum-volume bisector graph* $\mathcal{B}_{\text{min}}(\mathcal{P})$.

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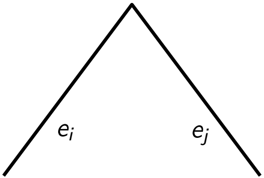
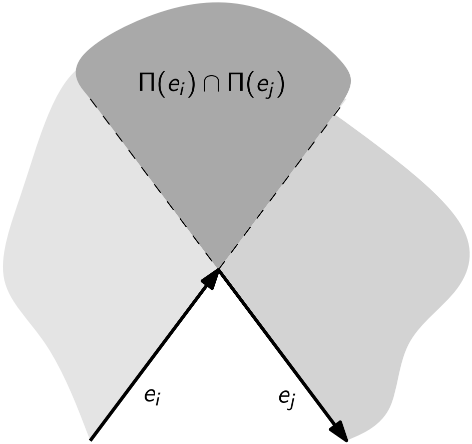
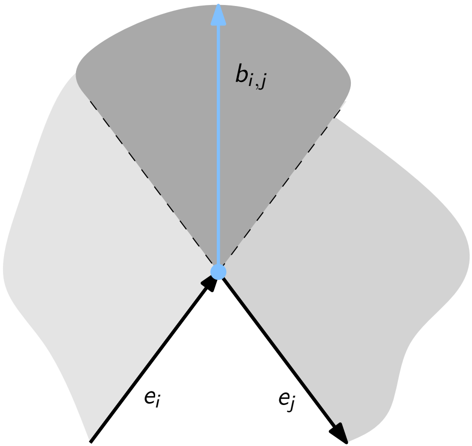
Two consecutive edges e_i, e_j of $\mathcal{P}$.



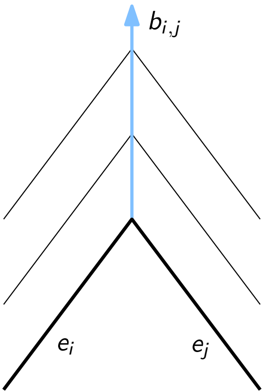
Edges of $\mathcal{P}$ are oriented. A half plane $\Pi(e)$ that starts at the supporting line $\ell(e)$ of an edge spans to its left. $\Pi(e)$ overlaps locally with the interior of $\mathcal{P}$.



A bisector $b_{i,j}$ spans from the intersection of the supporting line of two edges into their common interior.

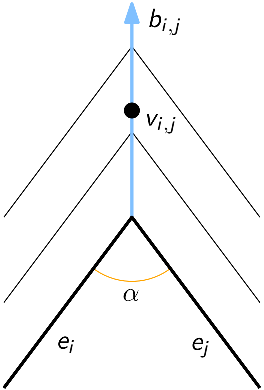


Wavefront propagation of e_i and e_j.



Wavefront propagation of e_i and e_j. A wavefront edge moves at unit speed (self parallel). The speed $s(v)$ of a wavefront vertex v depends on the angle between the supporting lines forming its bisector[3].

$$s(v_{i,j}) = \frac{1}{\sin(\alpha/2)}$$

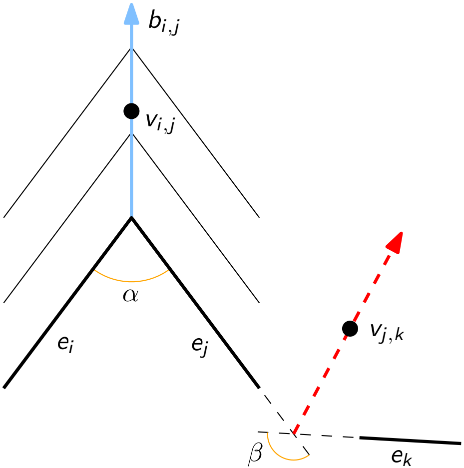


3. Siu-Wing Cheng and Antoine Vigneron. Motorcycle Graphs and Straight Skeletons. In *Proc. 13th Symposium on Discrete Algorithms*, 2002

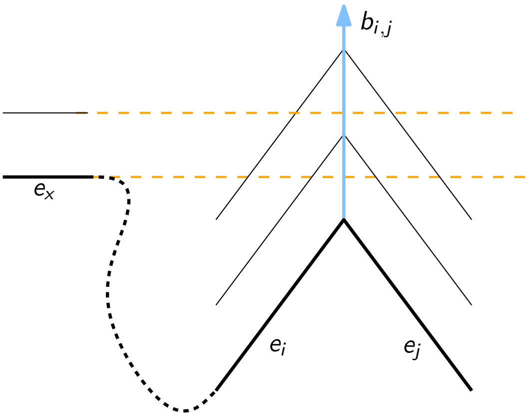
Every bisector defines a vertex that has a starting point and associated speed. In case such a vertex is not part of the wavefront we call it *stealth vertex*.

$$s(v_{i,j}) = \frac{1}{\sin(\alpha/2)}$$

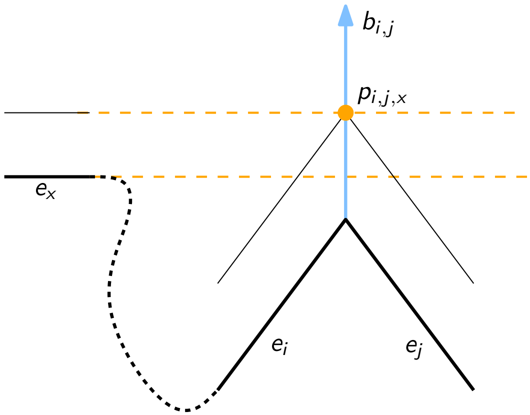
$$s(v_{j,k}) = \frac{1}{\sin(\beta/2)}$$



Another input edge e_x of $\mathcal{P}$.

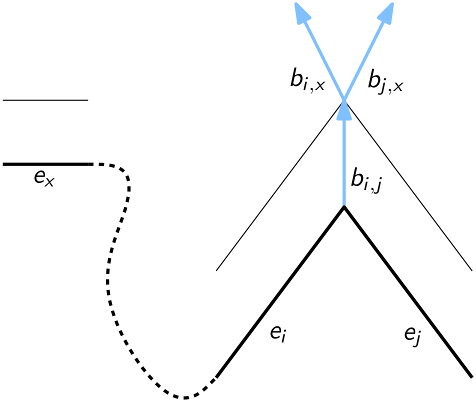


At some point $p_{i,j,x}$ is the wavefront vertex incident with the supporting line from the wavefront edge of e_x.

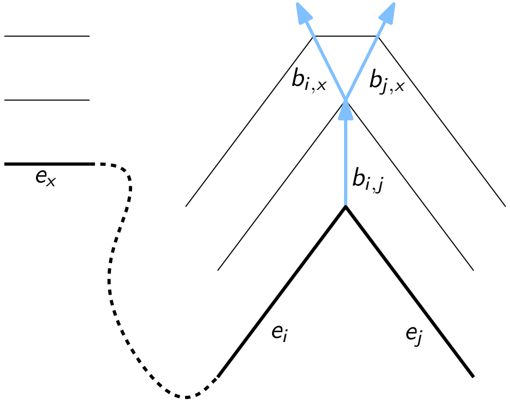


Create Event

At some point $p_{i,j,x}$ is the wavefront vertex incident with the supporting line from the wavefront edge of e_x. The three bisectors meet at that point as well.



The wavefront changes: an additional edge e is created, and e is parallel to the wavefront edge of e_x. The two wavefront vertices on $b_{i,x}$ and $b_{j,x}$ are both reflex.



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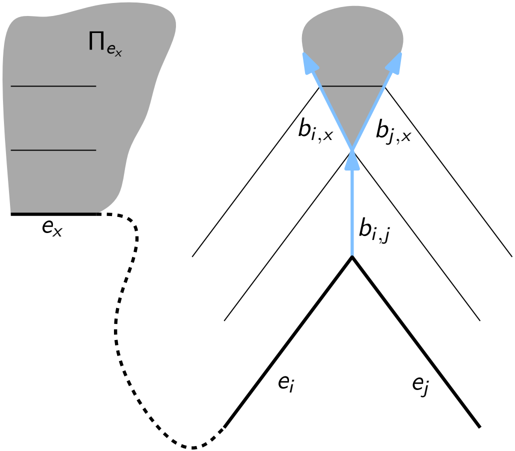
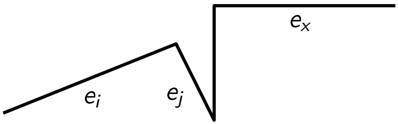
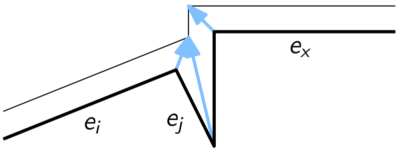


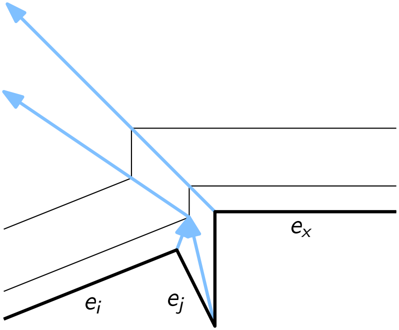
Consecutive edges along a polygon boundary.



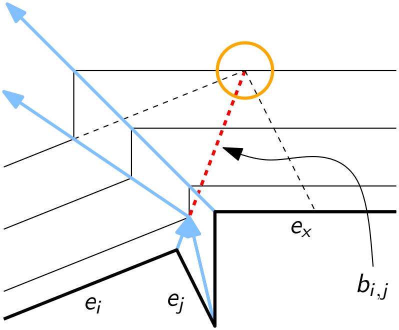
Consecutive edges along a polygon boundary. Wavefront propagation on the first (edge) event.



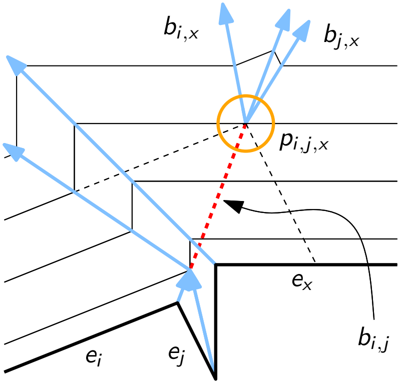
Consecutive edges along a polygon boundary. Wavefront propagation on the first (edge) event. Wavefront propagation continues.



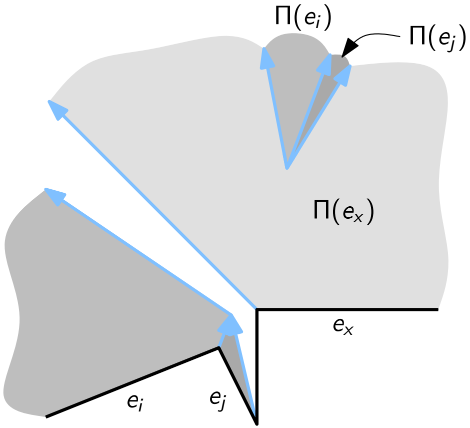
The stealth vertex $v_{i,j}$ becomes incident with the wavefront edge originating from e_x at point $p_{i,j,x}$.



The stealth vertex $v_{i,j}$ becomes incident with the wavefront edge originating from e_x at point $p_{i,j,x}$. Three arcs start at this point and create two new facets.



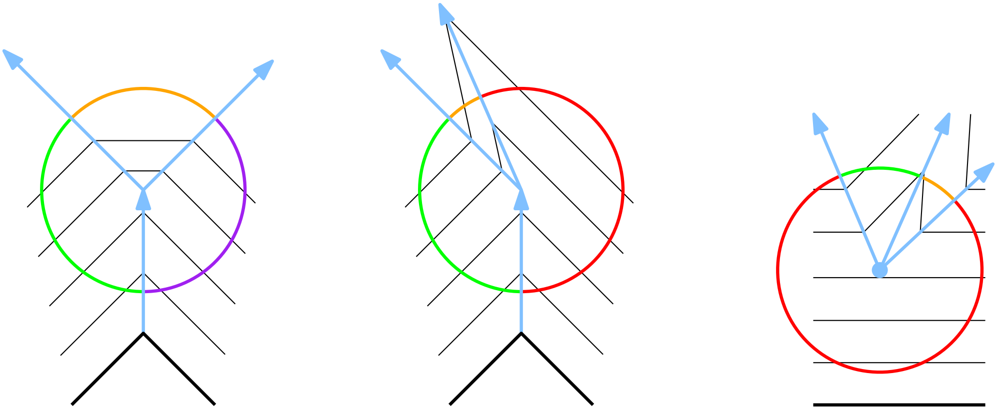
The stealth vertex $v_{i,j}$ becomes incident with the wavefront edge originating from e_x at point $p_{i,j,x}$. Three arcs start at this point and create two new facets. One of these facets lies in the plane $\Pi(e_i)$ and one in $\Pi(e_j)$.



Accelerating/Decelerating Create Event

Lemma

A small disc c centered around a create event p is partitioned into three wedges by the three arcs incident at p. If one wedge has an angle greater than π it involves a wavefront vertex, starting at p, that moves faster than the wavefront vertex which ends at p.



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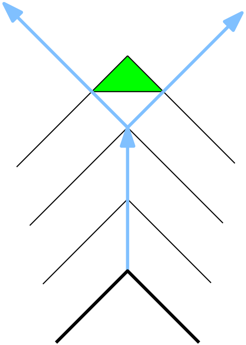
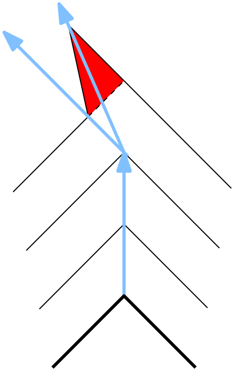
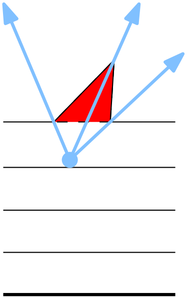


Decelerating



Accelerating



Accelerating

ALGORITHM

COMPLEXITY

- The wavefront propagation is used both to compute $\mathcal{B}_{\text{min}}(P)$ and $\mathcal{B}_{\text{max}}(P)$.
- The complexity is dominated by the computation of the create events.
- One create event takes $\mathcal{O}(n \log n)$ time to compute and enqueue.
- There can be up to $\mathcal{O}(n^2)$ create events.

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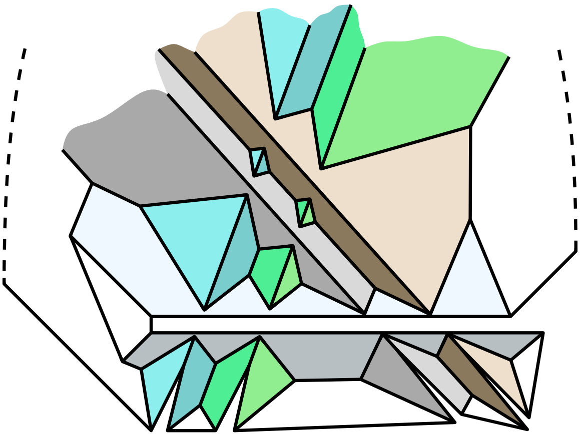
The overall complexity to compute $\mathcal{B}_{\min}(P)$ or $\mathcal{B}_{\max}(P)$ is in $\mathcal{O}(n^3 \log n)$.

End

Thanks for your attention!

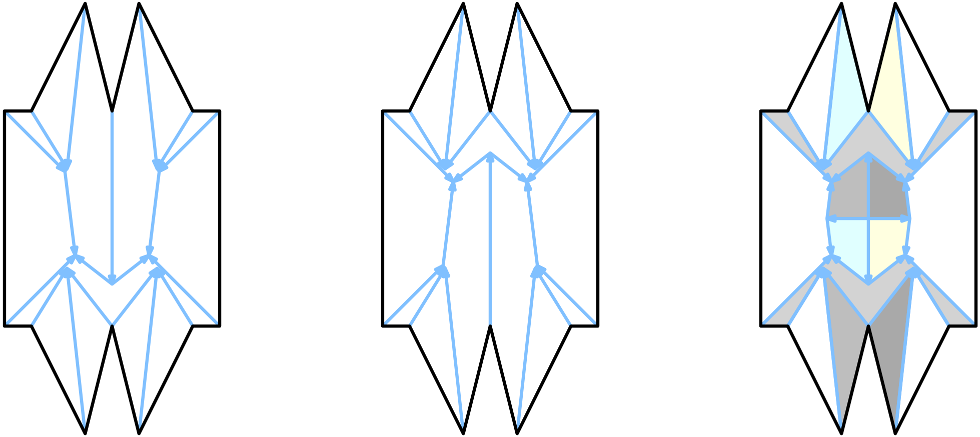
$\mathcal{B}_{\text{min}}$ / $\mathcal{B}_{\text{max}}$ Properties



Lemma

The number of facets $\mathcal{B}_{\text{min}}$ and $\mathcal{B}_{\text{max}}$ can have is in $\mathcal{O}(n^2)$.

$\mathcal{B}_{min}$ / $\mathcal{B}_{max}$ PROPERTIES



LEMMA

The upper envelope of two natural roofs is not necessarily a natural roof.

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